Clock analysis

Viktor Engelmann

January 23, 2009

Abstract

In tenth grade I wrote a program to determine the times of the day at wich the short hand and long hand of the clock have a given angle α to each other. Lacking math skills, I made the program try out all 1440 minutes of a day and compare the angles at the given times. Today I have the neccessary math knowledge, to solve this Problem properly.

1 Calculating the time to a given angle

Observe that every minute the long hand of the clock turns by $\frac{360°}{60} = 6°$. The short hand turns just by $\frac{360°}{12 \cdot 60} = 0.5°$ per minute and moves $\frac{360°}{12} = 30°$ per hour. So if it is h o'clock and m minutes, the hands have the angles

$$\begin{aligned} s(h,m) &= 30h + 0.5m \\ l(h,m) &= 6m \end{aligned}$$

The angle between the hands is

$$\begin{aligned} \alpha &= s(h,m) - l(h,m) \\ &= 30h + 0.5m - 6m \\ \Leftrightarrow \alpha - 30h &= (0.5 - 6)m \\ \Leftrightarrow \frac{\alpha - 30h}{-5.5} &= \frac{30h - \alpha}{5.5} = m \end{aligned}$$

This means that to determine the times of the day we simply have to replace h by the availiable hours $\{0, 1, 2, ..., 11\}$ and calculate the minute m for this particular hour.
Example: the hands have angle $\alpha = 90°$ at

$$\begin{aligned} m_0 = \frac{30 \cdot 0 - 90}{5.5} &= -16.\overline{36} \Rightarrow 11 : 43 : 38.\overline{18} \\ m_1 = \frac{30 \cdot 1 - 90}{5.5} &= -10.\overline{90} \Rightarrow 00 : 49 : 05.\overline{45} \\ m_2 = \frac{30 \cdot 2 - 90}{5.5} &= -5.\overline{45} \Rightarrow 01 : 54 : 32.\overline{72} \\ m_3 = \frac{30 \cdot 3 - 90}{5.5} &= 0 \Rightarrow 03 : 00 : 00 \\ &... \end{aligned}$$

Consider that for example 2 o'clock $-5.\overline{45}$ minutes is

$$\begin{aligned} & 1 : (60 - 5.\overline{45}) \\ = & \quad 1 : 54.\overline{54} \\ = & \quad 1 : 54 + 0.\overline{54} \\ = & \quad 1 : 54 : 60 \cdot 0.\overline{54} \\ = & \quad 1 : 54 : 32.\overline{72} \end{aligned}$$

2 Calculating the repeating time of an angle

Now I was intrigued and wanted to know after what timespan the hands would have the same angle again. At hour $(h+1)$ the associated minute would have changed by Δ minutes:

$$\begin{aligned} \frac{30(h+1)-\alpha}{5.5} &= \Delta + \frac{30h-\alpha}{5.5} \\ \Leftrightarrow \frac{30(h+1)-\alpha}{5.5} - \frac{30h-\alpha}{5.5} &= \Delta \\ \Leftrightarrow \frac{30h+30-30h-\alpha+\alpha}{5.5} &= \Delta \\ \Leftrightarrow \frac{30}{5.5} &= 5.\overline{45} = \Delta \end{aligned}$$

$5.\overline{45}$ minutes equal 5 minutes and $27.\overline{27}$ seconds, so every $1 : 05 : 27.\overline{27}$ hours the hands have the same angle again - independent of the angle!

Note: $1 : 05 : 27.\overline{27}$ hours are one eleventh of 12 hours.

3 Angle-alteration by 1°

Looking for more applications of the above results I calculated in which timespan c the angle between the hands changes by exactly 1°.
Let $m = \frac{30h-\alpha}{5.5}$.

$$\begin{aligned} \frac{30h-\alpha+1}{5.5} &= m + c \\ \Leftrightarrow \frac{30h-\alpha+1}{5.5} &= \frac{30h-\alpha}{5.5} + c \\ \Leftrightarrow 30h-\alpha+1 &= 30h-\alpha+5.5c \\ \Leftrightarrow 1 &= 5.5c \\ \Leftrightarrow c &= \frac{1}{5.5} min = 10.\overline{90}\text{sec.} \end{aligned}$$

4 Smiling times

Years after finding these results I learned from a TV quiz-show, that in advertisements for clocks, the clocks always show the time 1 : 50, because the clock "smiles", which increases the sales figures. Obviously at $1:50$ the angles of the hands aren't symmetrical, because the long hand has $(360-60)^\circ$, but the short hand only has $(60-10*0.5)^\circ = 55^\circ$, which caused me to think about the question, at which times the angles of the short hand and the long hand are symmetrical (so $s(h,m) = 360 - l(h,m)$). This is the case, iff

$$\begin{aligned} 6m &= 360 - (30h + 0.5m) \\ \Leftrightarrow 12m &= 720 - 60h - m \\ \Leftrightarrow 13m &= 720 - 60h \\ \Leftrightarrow m &= \frac{720 - 60h}{13} \end{aligned}$$

using this result, we can now explicitly calculate the times, at which the hands are symmetrical:

h	m	default representation
0	55.38	0:55:23.0769
1	50.76	1:50:46.1538
2	46.15	2:46:09.2307
3	41.54	3:41:32.3076
4	36.92	4:36:55.3846
5	32.3	5:32:18.4615
6	27.69	6:27:41.5384
7	23.08	7:23:04.6153
8	18.46	8:18:27.6923
9	13.85	9:13:50.7692
10	9.23	10:09:13.8461
11	4.62	11:04:36.9230
12	0	12:00:00

Note: the difference between one of these times, and the next, is always $55:23.0769$ minutes, which is a thirteenth of 12 hours.